MATH5051 Measure Theory and Functional Analysis

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FL2024 with Prof. Henri Martikainen

Contents

Logistics.

Office Hours: Tuesday 4-5, Thursday 1:30-2:30. Cupplies I 202.

Homework: Weekly, collected Thursday. (First 9/5)

Exams: One midterm. *Thursday October 17th (TBD.)*

1 Fundamentals

1.1 Set Theory

Let X be some ambient space and $\mathcal{F}$ be a collection of subsets $A \subset X$. For notation we write

$$\bigcup_{\alpha \in \mathcal{A}} A_\alpha = \bigcup_{A \in \mathcal{F}} A = \{x \in X : x \in A \text{ for some } A \in \mathcal{F}\}$$

$$\bigcap_{\alpha \in \mathcal{A}} A_\alpha = \bigcap_{A \in \mathcal{F}} A = \{x \in X : \forall A \in \mathcal{F}\}$$

Note that most of the time $\mathcal{F}$ would be countable. Otherwise, for example, $\mathbb{R}^d = \cup_{x \in \mathbb{R}^d} \{x\}$ would be zero measure.

1.2 Countable Sets

$\mathcal{F}$ is **countable** if $\mathcal{F} = \emptyset$ or $\exists$ injection $\phi : \mathcal{F} \to \mathbb{N}$, and note that countable union of countable sets is countable.

For **set complements**, the course prefers $X \setminus A$ rather than A^c

Recall that **De Morgan's Laws** are given as

$$X \setminus \bigcup_{A \in \mathcal{F}} A = \bigcap_{A \in \mathcal{F}} (X \setminus A)$$
$$X \setminus \bigcap_{A \in \mathcal{F}} A = \bigcup_{A \in \mathcal{F}} (X \setminus A)$$

2 Abstract Measure Theory

Definition. The **power set** $\mathcal{P}(X)$ is the set of all subsets of X.

Often, a measure cannot quite act on all elements of $\mathcal{P}(X)$. Thus, we limit this to σ-algebras.

Definition. A collection $\mathcal{F} \subset \mathcal{P}(X)$ is a **σ-algebra** if

1. $\emptyset \in \mathcal{F}$.
2. $X \setminus E \in \mathcal{F},\ \forall E \in \mathcal{F}$.
3. If $E_1, E_2, ... \in \mathcal{F}$, then countable union $\bigcup_{i=1}^{\infty} E_i \in \mathcal{F}$.

Remark: If $E_1, ... \in \mathcal{F}$, then $\bigcap E_i = X \setminus \bigcup (X \setminus E_i) \in \mathcal{F}$

2.1 Measure Spaces

Definition. Let $\mathcal{F}$ be a σ-algebra on a set X. A function $\mu : \mathcal{F} \to [0, \infty]$ is a **measure** if

1. $\mu(\emptyset) = 0$
2. *Countable Additivity:* For disjoint and countable $A_1, A_2, ... \in \mathcal{F}$, it is the case that

$$\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mu(A_i)$$

We call the tuple $(X, \mathcal{F}, \mu)$ a **measure space**. Sometimes $(X, \mathcal{F})$ is called a **measurable space**

Measures satisfy

1. *Monotonicity*: If $A \subset B$, $A, B \in \mathcal{F}$, then

$$\mu(B) = \mu(A \cup (B \setminus A)) = \mu(A) + \mu(B \setminus A) \implies \mu(A) \leq \mu(B)$$

2. *Subadditivity*: Let $A_1, A_2, ... \in \mathcal{F}$. Write $\cup A_i = \cup B_i$, where $B_i \in \mathcal{F}, B_i \subset A_i$, and B_i disjoint. This can be constructed with $B_1 := A_1, B_2 = A_2 \setminus B_1, B_n = A_n \setminus \cup_{i<n} B_i$, so therefore $\mu(\cup A_i) = \mu(\cup B_i) = \sum \mu(B_i) \leq \sum \mu(A_i)$, where the last inequality follows from monotonicity.
3. $\mu(B \setminus A) = \mu(B) - \mu(A)$ only when $A, B \in \mathcal{F}$ and $\mu(A) < \infty$

Remark: Subadditivity is often very useful.

Example. Suppose we throw two dices. The natural event space is $X = \{(i, j) : i, j \in \{1, ..., 6\}\}$. In this case, the sigma algebra can be entirety of $\mathcal{P}(X)$. One can then naturally define $\mu(A) = |A|/36$.

2.2 Outer Measures

Definition. We say $\mu : \mathcal{P}(X) \to [0, \infty]$ is an **outer measure** if

1. $\mu(\emptyset) = 0$
2. $A \subset B \subset X \implies \mu(A) \leq \mu(B)$
3. If $A = \cup_i A_i, A_i \subset X$, then $\mu(A) \leq \sum \mu(A_i)$

Now, if we have an outer measure, we can to find a σ-algebra $\mathcal{F}$ such that $\mu|_{\mathcal{F}} : \mathcal{F} \to [0, \infty]$ is a measure. Thus, we have the following definition of measurable sets:

Definition. Given an outer measure μ on X, we say $E \subset X$ is μ-measurable if $\forall A \subset X$, we have

$$\mu(A) = \mu(A \cap E) + \mu(A \setminus E)$$

We also write the collection of measurable sets as

$$\mathcal{M}_\mu = \mathcal{M}_\mu(X) = \{E \subset X : \mu - \text{measurable}\}$$

Remark: Note that due to subadditivity, it suffices to check that $\mu(A) \geq \mu(A \cap E) + \mu(A \setminus E) \forall A \subset X$ with $\mu(A) < \infty$.

Example. Define $\mu : \mathcal{P}(X) \to \{0, 1\}$ by $\mu(\emptyset) = 0$ and $\mu(A) = 1, A \neq \emptyset$.

Here, monoticity and subadditivity clearly holds (check notes for detail.) We claim that $\mathcal{M}_\mu = \{\emptyset, X\}$. Suppose there is $\phi \neq A \neq X$. Then $\mu(X) = 1 < 2 = \mu(A) + \mu(X \setminus A) \implies A \notin \mathcal{M}_\mu$.

Theorem. Let μ be an outer measure on X. Then $\mathcal{M}_\mu$ is a σ-algebra, and if $E_1, E_2, ... \in \mathcal{M}_\mu$ are *disjoint*, then

$$\mu\left(\bigcup E_i\right) = \sum \mu(E_i)$$

Proof. Claim 1: If $\mu(E) = 0$ satisfies $E \in \mathcal{M}_\mu$; inparticular, $\emptyset \in \mathcal{M}_\mu$.

Notice that $\forall A \subset X$, $\mu(A) \geq \mu(A \setminus E) = \mu(A \cap E) + \mu(A \setminus E)$ by monoticity and $\mu(A \cap E) = 0$. Thus, $E \in \mathcal{M}_\mu$

Claim 2: $E \subset X$ measurable $\iff$ $X \setminus E$ measurable. This is obvious from definition.

Claim 3: $E_1, .., E_n \in \mathcal{M}_\mu \implies \bigcup_{i=1}^N E_i \in \mathcal{M}_\mu$.

It suffices to prove $N = 2$. Let $A \subset X$ be a test set for measurability. Since $E_1 \in \mathcal{M}_\mu$, $\mu(A) = \mu(A \cap E_1) + \mu(A \setminus E_1)$. We can expand this again into

$$\begin{aligned}
\mu(A) &= \mu(A \cap E_1) + \mu(A \cap (X \setminus E_1) \cap E_2) + \underbrace{\mu(A \cap (X \setminus E_1) \setminus E_2)}_{=\mu(A \setminus (E_1 \cup E_2))} \\
&\geq \mu((A \cap E_1) \cup (A \cap (X \setminus E_1) \cap E_2)) + \mu(A \setminus (E_1 \cup E_2)) \\
&= \mu(A \cap (E_1 \cup E_2)) + \mu(A \setminus (E_1 \cup E_2)) \implies \boxed{E_1 \cup E_2 \in \mathcal{M}_\mu}
\end{aligned}$$

Claim 4: $E_1, ..., E_n \in \mathcal{M}_\mu \implies \bigcap_{i=1}^N E_i \in \mathcal{M}_\mu$. This is clear since $\bigcap_{i=1}^N E_i = X \setminus \bigcup_{i=1}^N (X \setminus E_i)$.

Claim 5: If E, F measurable, then $E \setminus F$ measurable.

Claim 6: If $E_1, ..., E_N \in \mathcal{M}_\mu$ *disjoint*, then $\forall A \subset X$ need not measurable,

$$\mu\left(A \cap \bigcup_{i=1}^N E_i\right) = \sum_{i=1}^N \mu(A \cap E_i)$$

Again, it suffices to show $N = 2$.

$$\begin{aligned}\mu(A \cap (E_1 \cup E_2)) &= \mu(A \cap (E_1 \cup E_2) \cap E_1) + \mu(A \cap (E_1 \cup E_2) \setminus E_1) \\ &= \mu(A \cap E_1) + \underbrace{\mu(A \cap E_2)}_{\text{since disjoint}}\end{aligned}$$

Claim 7: If $E_1, E_2, ... \in \mathcal{M}_\mu$, then $\bigcup_{i=1}^N E_i \in \mathcal{M}_\mu$. If they are disjoint, then

$$\mu\left(\bigcup_{i=1}^\infty E_i\right) = \sum_{i=1}^\infty \mu(E_i)$$

First, let $E_1, E_2, ... \in \mathcal{M}_\mu$. Use the trick of setting equal disjoint unions, where $\cup F_i = \cup E_i$. Namely, $F_1 = E_1, F_n = E_n \setminus (F_1 \cup \cdots \cup F_{n-1})$. By previous claims, we have established that such $F_i \in \mathcal{M}_\mu$. Also since $F_i \subset E_i \forall i, F_i$ are all disjoint as well.

Define also $S_k := \cup_{i=1}^k F_i \in \mathcal{M}_\mu$, $S_k \subset E$. To show $E \in \mathcal{M}_\mu$, fix $A \subset X$. Then, $S_k \in \mathcal{M}_\mu \implies$

$$\begin{aligned}\mu(A) = \quad & \underbrace{\mu(A \cap S_k)}_{\text{Claim 6},= A \cap \bigcup_{i=1}^k F_i} \quad + \mu(A \setminus S_k) \\ & \geq \sum_{i=1}^k \mu(A \cap F_i) + \mu(A \setminus E)\end{aligned}$$

Then, let $k \to \infty$ and using subadditivity,

$$\begin{aligned}\mu(A) &\geq \sum_{i=1}^\infty \mu(A \cap F_i) + \mu(A \setminus E) \qquad (*) \\ &\geq \mu\left(\underbrace{\bigcup_{i=1}^\infty A \cap F_i}_{= A \cap \bigcup F_i}\right) + \mu(A \setminus E) \\ &= \mu(A \cap E) + \mu(A \setminus E) \\ &\therefore E \in \mathcal{M}_\mu\end{aligned}$$

Notice that by $(*)$ with $A = E$, it gives $\mu(E) \geq \sum_{i=1}^{\infty} \mu(\underbrace{E \cap F_i}_{=F_i}) + \underbrace{\mu(E \setminus E)}_{=0} = \sum_{i=1}^{\infty} \mu(F_i)$.

If $E_1, E_2, ...$ disjoint, then $E_i = F_i$. ■

The following presents a useful way to construct outer measure.
Lemma. Suppose $\mathcal{S} \subset \mathcal{P}(X), \emptyset \in \mathcal{S}$. Let $h : S \to [0, \infty]$ be a function with $h(\emptyset) = 0$. For $A \subset X$, define

$$\mu(A) := \inf \left\{ \sum_{i=0}^{\infty} h(S_i); A \subset \bigcup_{i=1}^{\infty} S_i, S_i \in \mathcal{S} \right\}$$

Then, such μ is an outer measure. (By convention, $\inf \phi = \infty$, meaning $\mu(A) = \infty$ if $\nexists$ S_i s.t. $A \subset \cup S_i$.)

2.3 Lebesgue Measure

Let $X = \mathbb{R}^d$. Let $S = \{R = I_1 \times \cdots \times I_d$ rectangles with sides parallel to axis$\}$. If $R = \prod_{i=1}^{d} [a_i, b_i]$, define $h(R) = vol(R) = \prod_{i=1}^{d} (b_i - a_i)$. Then, μ given by the lemma above is the **Lebesgue Outer Measure**, defined on $\mathcal{P}(\mathbb{R}^d)$. In particular, we denote $\mu(A) = |A|$.

Using the result from last time, $\exists \sigma$-algebra $\mathcal{M}_{\mu^d} = LEB(\mathbb{R}^d)$ called the **Lebesgue measurable sets**, where $|\cup A_i| = \sum |A_i|$ whenever $A_i \in LEB(\mathbb{R}^d)$, $A_i \cap A_j = \emptyset, \forall i \neq j$.

Note that $LEB(\mathbb{R}^d) \neq \mathcal{P}(\mathbb{R}^d)$. See appendix for details.

2.4 Convergence Results for Measures

Theorem. Let $(X, \mathcal{F}, \mu)$ be a measure space.

1. If $A_1 \in \mathcal{F}$ and $A_1 \subset A_2 \subset ...$, then $\mu\left(\cup_{i=1}^{\infty} A_i\right) = \lim_{i \to \infty} \mu(A_i)$.
2. Let $A_i \in \mathcal{F}, A_1 \supset A_2 \supset \cdots$ AND $\mu(A_i) < \infty$, then $\mu(\cap_{i=1}^{\infty} A_i) = \lim_{i \to \infty} \mu(A_i)$.

Proof (1). Let $A_0 = \emptyset$, then $\cup_{i=1}^{\infty} A_i = \cup_{i=1}^{\infty} (A_i \setminus A_{i-1})$ since they are disjoint union of elements of $\mathcal{F}$, and so

$$\begin{aligned}
\mu(\cup A_i) &= \sum_{i=1}^{\infty} \mu(A_i \setminus A_{i-1}) \\
&= \lim_{N \to \infty} \sum_{i=1}^{N} \mu(A_i \setminus A_{i-1}) \\
&= \lim_{N \to \infty} \mu\left(\bigcup_{i=1}^{N} A_i \setminus A_{i-1}\right) \\
&= \lim_{N \to \infty} \mu(A_n)
\end{aligned}$$

■

Proof (2). Define $A_1 \setminus A_i =: B_i$.

$$\begin{aligned}\mu\left(\cap_{i=1}^{\infty} A_i\right) &= \mu(A_1) - \mu\left(A_1 \setminus \bigcap_{i=1}^{\infty} A_i\right) \\ &= \mu(A_1) - \underbrace{\lim_{j\to\infty} \mu(B_j)}_{=\mu(A_1)-\mu(A_j)} \\ &= \lim_{j\to\infty} \mu(A_j)\end{aligned}$$

■

Briefly about *Concrete Measures.* We usually say something holds $\mu -$ **a.e.** (almost everywhere) if $N \in \mathcal{F}$ s.t. $\mu(N) = 0$, and the property bolds in $X \setminus N$. For instance, $f(x) < \infty$ **a.e.** w.r.t lebesgue measure, if $|\{f(x) = \infty\}| = 0$.

Abstractly, there could be a problem. Even if $N \in \mathcal{F}, \mu(N) = 0$, there can exists $N' \subset N$ such that $N' \notin \mathcal{F}$. We say $(X, \mathcal{F}, \mu)$ is **complete** if the above cannot happen. In general, we can always complete any measure space.

Notation: $\mathcal{N} := \{N \in \mathcal{F} : \mu(N) = 0\}$. $\mathcal{N}' := \{N' \subset X : \exists N \in \mathcal{N}$ such that $N' \subset N\}$. So, it is complete $\iff \mathcal{N} = \mathcal{N}'$.

Lemma. Suppose $(X, \mathcal{F}, \mu)$ is a measure space. Then

$$\bar{\mathcal{F}} := \{F \cup N' : F \in \mathcal{F}, N' \in \mathcal{N}'\}$$

is a σ-algebra, and $\exists!$ complete measure space $(X, \bar{\mathcal{F}}, \bar{\mu})$ such that $\bar{\mu}(F) = \mu(F), \forall F \in \mathcal{F}$. Notice that also the μ outer measusre $(X, \mathcal{M}_\mu, \mu|_{\mathcal{M}_\mu})$ is complete.

Proof. First we show that $\bar{\mathcal{F}}$ is really a σ-algebra. Clearly, $\emptyset \in \bar{\mathcal{F}}$. Notice

$$\bigcup_{i=1}^{\infty}(F_1 \cup N_i') = \underbrace{\bigcup_{i=1}^{\infty} F_i}_{\in\mathcal{F}} \cup \bigcup_{i=1}^{\infty} N_i', \text{ and } \bigcup N_i' \subset \underbrace{\bigcup N_i}_{\in\mathcal{F}, \text{ and } \mu(\cup N_i) \le \sum \mu(N_i) = 0}$$

$\bigcup(F_i \cup N_i') \in \overline{\mathcal{F}}$.

Finally, if $F \cup N' \in \bar{\mathcal{F}}$, then $X \setminus (F \cup N')$. In particular,

$$X \setminus (F \cup N') = [X \setminus \underbrace{(F \cup N)}_{\in\mathcal{F}}] \cup \underbrace{[N \setminus (N' \cup F)]}_{\subset N} \in \bar{\mathcal{F}}$$

Hence, $\bar{\mathcal{F}}$ is a σ-algebra.

Then, for $F \cup N' \in \bar{\mathcal{F}}$ and define $\bar{\mu}(F \cup N') := \mu(F)$.

A natural question arises: Well-defined? (if $F_1 \cup N_1' = F_2 \cup N_2'$, is $\mu(F_1) = \mu(F_2)$). Indeed, $F_1 \subset F_1 \cup N_1' = F_2 \cup N_2' \subset F_2 \cup N_2 \implies \mu(F_1) \leq \mu(F_2) + \mu(N_2) = \mu(F_2)$, and reverse to reach equality.

■

2.5 Borel Algebra

Definition. $\Gamma \subset \mathcal{P}(X)$, then

$$\sigma(\Gamma) := \bigcap \{\mathcal{F}\sigma - \text{algebra}, \mathcal{F} \supset \Gamma\}$$

is the σ-algebra generated by Γ, the "smallest σ-algebra containing Γ"

If X is a *topological space*, i.e., open sets make sense in X, then the **Borel Algbera** is

$$BOR(X) := \sigma\{\text{All open } V \subset X\}$$

For instance, recall $LEB(\mathbb{R}^d)$. We can prove $V \subset \mathbb{R}^d$ open $\implies V \in LEB(\mathbb{R}^d)$. This implies that

$$BOR(\mathbb{R}^d) \subset LEB(\mathbb{R}^d)$$

Actually, $BOR(\mathbb{R}^d) \subsetneq LEB(\mathbb{R}^d)$. In addition, $(\mathbb{R}^d, BOR(\mathbb{R}^d), m_d)$ is not a complete space, while $(\mathbb{R}^d, LEB(\mathbb{R}^d), m_d)$ is complete.

Example. In $\mathbb{R}$, half open and closed intervals are closed, since

$$[a, b) = \bigcup_{i=1}^{\infty} [a, b - 1/i] \in BOR(\mathbb{R})$$

3 Measurable Functions

Definition. The **extended reals** are defined as $\dot{\mathbb{R}} := \mathbb{R} \cup \{\pm\infty\}$. Let $(X, \mathcal{F}, \mu)$ be a measure space.

Definition. A function $f : X \to \mathbb{R}$ is **measurable** ($\mathcal{F}$)-measurable if

$$f^{-1}[-\infty, a) = \{x \in X; f(x) < a\} \in \mathcal{F}, \quad \forall a \in \mathbb{R}$$

In particular, this definition automatically allows for more complicated preimages to be measurable, such as $\{f(x) = a\}, \{f > a\}, \{f \geq a\}, \{f \leq a\}$. In fact, $f^{-1}B \in \mathcal{F}, \forall B \in BOR(\mathbb{R})$.

Lemma. Let $f : X \to \dot{\mathbb{R}}$ and

$$\Gamma_f = \{M \subset \dot{\mathbb{R}} : f^{-1}M \in \mathcal{F}\}$$

Then, Γ_f is a σ-algebra.

Proof. $M \in \Gamma_f \implies f^{-1}M \in \mathcal{F} \implies f^{-1}(\dot{\mathbb{R}} \setminus M) = X \setminus f^{-1}M \in \mathcal{F}$.

Similarly, let $M_1, M_2, ... \in \Gamma_f \implies f^{-1} \cup M_i = \cup f^{-1}M_i \in \mathcal{F}$ ∎

Being measurable means the σ-algebra σ_f contains all intervals $[-\infty, a)$

Corollary. If $f : X \to \dot{\mathbb{R}}$ is *measurable*, then $BOR(\mathbb{R}) \subset \Gamma_f$, i.e., $f^{-1}B \in \mathcal{F}$. $\forall B \in BOR(\mathbb{R})$, also $f^{-1}\{\pm\infty\} \in \mathcal{F}$

Proof. First, to prove $BOR(\mathbb{R}) \subset \Gamma_f$, it is enough to prove $V \subset \mathbb{R}^d$ open $\implies V \in \Gamma_f$. (Lemma 6.1, later in the course).

Every open $V \subset \mathbb{R}$ is a (disjoint), countable union of half-open intervals $[a, b)$. So, $[a, b) \in \Gamma_f$ is enough.

Indeed, $[-\infty, a) \in \Gamma_f$, and $[a, b) = [a, \infty] \cap [-\infty, b) \in \Gamma_f$ ∎

Example. Let $f, g : X \to \mathbb{R}$ be measurable. $\{x \in X : f(x) = g(x)\} = \{x : f - g = 0\} = (f - g)^{-1}\{0\}$. So, this is measurable if $f - g$ is.

<u>Remark:</u> Compositions are the only operation where we need to be a bit careful with measurability. Suppose, for example, $f : \mathbb{R} \to \mathbb{R}$ is borel measurable (i.e. $f^{-1}(B) \in$ Boreal, $\forall B \in BOR(\mathbb{R})$) and $g : X \to \mathbb{R}$ is measurable. Then, $(f \circ g)^{-1}B = g^{-1}\underbrace{(f^{-1}(B))}_{\in BOR(\mathbb{R})}$ $\forall B \in BOR(\mathbb{R}) \implies f \circ g$ is $\mathcal{F}$-measurable.

Proposition. Let $f, g : X \to \mathbb{R}$ be real-valued measurable functions, and $\alpha \in \mathbb{R}$. Then $\alpha f, f + g, fg, f/g$ are measurable, assuming $g \neq 0$ for f/g.

Proof. 1. αf is trivial.

2. For $f+g$, notice that $\{f+g<a\}=\bigcup_{q,r\in\mathbb{Q},q+r<a}[\{f<q\}\cap\{g<r\}]$ is in $\mathcal{F}$, since is a *countable* union of elements in $\mathcal{F}$.

3. For fg, notice first that f^2 is measurble. Then we can use $fg=[(f+g)^2-f^2-g^2]/2$.

4. $f/g=f\cdot 1/g$.

■

Proposition. Let $f:X\to\mathbb{R}$ measurable $\implies$ $|f|^p$ is measurable $\forall p>0$.

Proof. $|f|^p=\phi\circ f,\phi(x):=|x|^p$. here, $\phi:\mathbb{R}\to\mathbb{R}$ is Bor($\mathbb{R}$) measurable since $\{\phi<a\}=\phi^{-1}[-\infty,a)$ is open (consider continuity.) Apply the composition remark. ■

Most critical for modern analysis: various limits of measurable functions are still measurable.

Lemma. if $f_j:X\to\dot{\mathbb{R}}$ is measurable, then $\sup_j f_j,\inf_j f_j,\limsup_j f_j,\liminf_j f_j$ are measurable.

Recall that for scalars $a_1,\ldots$, $\limsup a_i=\lim_{k\to\infty}\sup_{i\geq k}a_i=\inf_k\sup_{i\geq k}a_i$

Proof. Note: this applied for all a usually implies intersection.

$$\{\sup_j f_j\leq a\}=\bigcap_{j=1}^{\infty}\{\underbrace{f_j\leq a}_{\in\mathcal{F}}\}\in\mathcal{F}$$

Note that the inequality above only holds for $\leq$, not $<$.

And also $\inf_j f_j=-\sup_j(-f_j)\in\mathcal{F}$. Thus $\limsup_{j\to\infty}f_j=\inf_J\sup_{j\geq J}f_j$. ■

Corollary. If $f_j\to f$ pointwise and each f_j is measurable, then f is measurable.

Proof. $f=\lim_j f_j=\liminf f_j$ is measurable. ■

Lemma. Suppose $(X,\mathcal{F},\mu)$ is a *complete* measure space, $f:X\to\mathbb{R}$ measurable and $g(x)=f(x)$ for μ-a.e. X. Then, g is measurable.

Proof. Let $N\in\mathcal{F}$ such that $\mu(N)=0$ and $f(x)=g(x)\,\forall x\in X\setminus N$. Now, $\{g<a\}=\{x\in X\setminus N;f(x)<a\}\cup\{\underbrace{x\in N:g(x)<a}_{\in\mathcal{F}\text{ by completeness as a subset of }N}\}\in\mathcal{F}$. ■

Remark. Sometimes this is expresses as a class of equivalence functions $[f]=\{g:g=f,\mu-\ \textbf{a.e.}\}$.

3.1 Simple Functions

Definition. A **simple function** is a measurable function $s : X \to \mathbb{R}$ that takes only finitely many values.

Note. For this class we use 1_E to denote the indicator that equals 1 if $x \in E$, and 0 otherwise.

A function of the following form is *simple*:

$$\sum_{i=1}^{N} c_i 1_{E_i}, E_i \in \mathcal{F},\ c_1, ..., c_n \in \mathbb{R}$$

A simple function can be written in many ways like above, like $0 = 1_E - 1_E$. There is a particular caonical representation, however. Let s be simple, and $c_1, ..., c_n$ be its *distinct* values. Define

$$E_i := s^{-1}\{c_i\} = \{x \in X : s(x) = c_i\} \in \mathcal{F}$$

Then, $X = \cup E_i$ is a disjoint union and $s = \sum_{i=1}^{N} c_i 1_{E_i}$.

Example. For $s = 1_{[0,10]} + s_{[0,100]}$, the canonical representation is $s = 2 \cdot 1_{[0,10]} + 1 \cdot 1_{[10,100]} + 0 \cdot 1_{\text{otherwise}}$.

The following is key for this canonical representation:

Lemma. Let $f : X \to \dot{\mathbb{R}}$ be a measurable function and non-negative, $f \geq 0$. There $\exists$ simple s_j s.t. $0 \leq s(x) \leq s_{j+1}(x) \leq f(x)$, and $f(x) = \lim_{j\to\infty} s_j(x) \forall x \in X$.

Proof. Define

$$s_1(x) = \begin{cases} 0, & 0 \leq f(x) \leq 1 \\ 1, & f(x) \geq 1 \end{cases},\ s_2(x) = \begin{cases} 0, & 0 \leq f(x) \leq \frac{1}{2} \\ 1/2, & 1/2 \leq f(x) < 1 \\ 1, & 1 \leq f(x) < 3/2 \\ 3/2, & 3/2 \leq f(x) < 2 \\ 2, & f(x) \geq 2 \end{cases}$$

In general, we can write

$$s_j(x) = \begin{cases} \frac{i-1}{2^j}, & \frac{i-1}{2^j} \leq f(x) < \frac{i}{2^j}, i = 2, ..., j \cdot 2^j \\ j, & f(x) \geq j \end{cases}$$

This is clearly measurable since we have finite unions of preimages of half-closed intervals and $f^{-1}[j, \infty]$.

By construction, they are increasing and less than f.

Convergence: If $f(x) = \infty$, then $s_j(x) = j \to \infty$, as $j \to \infty$. If $f(x) < \infty$, then for all big enough j, we have $f(x) - 2^{-j} < s_j(x) \leq f(x)$. So, $f(x) = \lim_{j\to\infty} s_j(x)$. ∎

4 Integration Against a General Measure

4.1 Integral of non-negative simple function.

Definition. Suppose s is simple and non-negative, written in its standard form $\sum_{i=1}^{N} c_i 1_{E_i}$. Then, the μ-**integral** is defined as

$$\int_X s d\mu = \int_X s(x) d\mu(x) = \int s d\mu := \sum_{i=1}^{N} c_i \mu(E_i)$$

<u>Remark.</u>

1. Later, we see $s = \sum c_i 1_{E_i} \implies \int s d\mu = \sum c_i \mu(E_i)$ even if not the canonical representation.
2. Notice $\mu(E_i) = \infty$ is possible. In that case, we simply define $c_i \cdot \infty = \{0, \text{ if } c_i = 0;\ \infty, \text{ otherwise}\}$.

Definition. If $E \in \mathcal{F}'$ and $s \geq 0$ is simple, define

$$\int_E s\, d\mu := \int_X 1_E s\, d\mu$$

Lemma. Let $s, u \geq 0$ be simple with $\int s\, d\mu, \int u\, d\mu < \infty$. Let $\alpha \in \mathbb{R}$ such that $s_\alpha u \geq 0$. Then,

$$\int (s + \alpha u) d\mu = \int s\, d\mu + \alpha \int u\, d\mu$$

Remark. If $\alpha \geq 0$, no need to assume $\int s\, d\mu, \int u\, d\mu < \infty$.

Proof. Let us write the cannoical representations

$$s = \sum_{i=1}^{I} b_i 1_{B_i} \qquad u = \sum_{j=1}^{J} c_j 1_{C_j}$$

Let $d_1, ..., d_K$ be enumeration of the set $\{b_i + \alpha c_j : i = 1, ..., I, j = 1, ..., J\}$. Given $k \in \{1, ..., K\}$, let

$$\mathcal{F}_k = \{(i,j); b_i + \alpha c_j = d_k\}$$

Now, a cannonical represention is

$$s + \alpha u = \sum_{k=1}^{K} d_j 1_{D_k}, D_k := \bigcup_{(i,j)\in\mathcal{F}_k} \underbrace{(B_i \cap C_j)}_{\text{disjoint}} \implies$$

$$\begin{aligned}
\int (s+\alpha u)\,d\mu &:= \sum_{k=1}^{K} d_k \mu(D_k) = \sum_{k=1}^{K} d_k \sum_{(i,j)\in\mathcal{F}_k} \mu(B_i \cap C_j) \\
&= \sum_{k=1}^{K} \sum_{(i,j)\in\mathcal{F}_k} (b_i + \alpha c_j)\mu(B_i \cap C_j) \qquad (*) \\
&= \sum_{i=1}^{I} \sum_{j=1}^{J} (b_i + \alpha c_j)\mu(B_i \cap C_j) \\
&= \sum_{i=1}^{I} b_i \sum_{j=1}^{J} \mu(B_i \cap C_j) + \alpha \sum_{j=1}^{J} c_j \sum_{i=1}^{I} \mu(B_i \cap C_j).
\end{aligned}$$

where $(*)$ follows from the addivity of μ where we brough d_k inside the summation. Also, we have

$$\sum_{j=1}^{J} \mu(B_i \cap C_j) = \mu(\bigcup_{j=1}^{J} B_i \cap C_j) = \mu(B_i \cap \bigcup_{j=1}^{J} C_j) = \mu(B_i)$$

and similarly for C.

■

Lemma. If s, u are simple functions and $0 \le s \le u$, then

$$\int s\,d\mu \le \int u\,d\mu \qquad \text{(Integration Monotonicity)}$$

Proof.

$$0 \le \int \underbrace{(u-s)}_{\ge 0}\,d\mu = \int u\,d\mu - \int s\,d\mu$$

WLOG, $\int u\,d\mu < \infty$, and thus it follows that also $\int s\,d\mu$ is finite. ■

Lemma. [Independence of Represention] Let $s = \sum_{i=1}^{N} c_i 1_{Ei}$ be not necessarily the canonical representation, with $c_i \in [0, \infty)$ and E_i measurable. Then,

$$\int s\,d\mu = \sum_{i=1}^{N} c_i \mu(E_i)$$

Proof. $\int s\,d\mu = \sum_{i=1}^{N} c_i \int 1_{E_i}\,d\mu$, where $\int 1_{E_i} d\mu = \mu(E_i)$. ■

Remark. Suppose we are integrating a simple function over a measurable $A \subset X$. Let $0 \leq s = \sum_{i=1}^{N} c_i 1_{E_i}$ simple.

$$\int_A s\, d\mu \overset{\text{def}}{=} \int_A \mathbb{I}_A s\, d\mu = \int \sum_{i=1}^{N} c_i \mathbb{I}_A \mathbb{I}_{E_i}\, d\mu = \int \sum_{i=1}^{N} c_i \mathbb{I}_{A\cap E_i}\, d\mu \overset{lemma}{=} \sum_{i=1}^{N} c_i \mu(E_i \cap A)$$

In particular, if $\mu(A) = 0$, $\int_A s\, d\mu$.

Theorem. Fix a simple function $s \geq 0$. Then the function

$$A \mapsto \int_A s\, d\mu$$

from μ-measurable sets to $[0, \infty]$ is a measure.

Proof. $\mu(\emptyset) = 0 \implies \int_\emptyset s d\mu = 0$.

This must have also satisfy countable additivity: Let $A_1, A_2, ...$ be measurable and pairwise disjoint. We must show that $\int_A s\, d\mu = \sum_{i=1}^{\infty} \int_{A_i} s\, d\mu$.

Let $s = \sum_{i=1}^{N} c_i 1_{E_i}$. Then

$$\begin{aligned}
\int_A s\, d\mu &= \sum_{i=1}^{N} c_i \cdot \mu(E_i \cap A) \\
&= \sum_{i=1}^{N} c_i \sum_{j=1}^{\infty} \mu(E_i \cap A_j) && (\mu \text{ countably additive}) \\
&= \sum_{j=1}^{\infty} \sum_{i=1}^{N} c_i \mu(E_i \cap A_j) \\
&= \sum_{j=1}^{\infty} \int_{A_j} s\, d\mu
\end{aligned}$$

■

Corollary. Let $E_1 \subset E_2 \subset \cdots$ be measruable sets and $s \geq 0$ simple. Then,

$$\int_{\cup E_j} s\, d\mu = \lim_{j\to\infty} \int_{E_j} s\, d\mu.$$

Proof. Apply convergence results to the measure $\nu(A) := \int_A s\, d\mu$ ■

4.2 Integration of Non-negative Measurable Functions

Definition. Let $f \geq 0$ be a measurable function.

$$\int f \ d\mu := \sup\left\{\int s \, d\mu; 0 \leq s \leq f \text{ simple}\right\}$$

Remark. Notice that via this definition, this is a non-empty set since the zero function exists. Also by monotonicity of integration of simple functions, when $0 \leq f \leq g$, the sets $\{s : 0 \leq s \leq f, s \text{ simple}\} \subset \{s : 0 \leq s \leq g, s \text{ simple}\}$ so that $\int f \, d\mu \leq \int g \, d\mu$.

The *key* in lebesgue integration is its behavior under limits.

Theorem. [Monotone Convergence Theorem].
Suppose f_j are measurable and $0 \leq f_1 \leq f_2 \leq \cdots$. Let $f(x) = \lim_{j\to\infty} f_j(x)$. This limit exists because of increasing sequence, and automatically measurable, since we prove limit of measurable functions is measurable. Then,

$$\int \lim_{j\to\infty} f_j \, d\mu = \int f \, d\mu = \lim_{j\to\infty} \int f_j d\mu$$

Proof. $B = \lim_{j\to\infty} \int f_j \, d\mu$ exists since the integral is increasing by monotonicity. Set $A := \int f \, d\mu$, which is fine since f measurable. We want to claim that $A = B$.

Clearly, $A \geq B$ since $f_j \leq f \forall j \implies \int f_j \, d\mu \leq \int f \, d\mu$.

For $B \geq A$, it suffices to prove that $\forall \delta \in (0,1), B \geq \delta A$: Let $0 \leq s \leq f$ be simple function, and we prove $\delta \int s \, d\mu \leq B$, since taking supremum over s yields $\delta A \leq B$.

Consider sets $E_j := \{\delta s \leq f_j\}$. Notice that E_j increasing since $f_j \leq f_{j+1}$. Also $\bigcup E_j = X$: fix $x \in X$. If $s(x) = 0$, then $\delta s(x) = 0$, then $\delta s(x) = 0 \leq f_1(x) \implies x \in E_1$. Otherwise if $s(x) > 0$, $s(x) \leq f(x)$ by definition of f, so $\delta s(x) < f(x)$. As $f_j(x) \to f(x)$, we must have $f_j(x) \geq \delta s(x)$ for some j, so $x \in E_j$. This proves that $\bigcup E_j = X$.

Thus applying corollary above,

$$\begin{aligned} \delta \int s \, d\mu &\overset{coro.}{=} \lim_{j\to\infty} \int_{E_j} \delta s \, d\mu = \lim_{j\to\infty} \int \mathbb{I}_{E_j} \delta s \, d\mu \leq \lim_{j\to\infty} \int \mathbb{I}_{E_j} f_j \, d\mu \\ &\leq \lim_{j\to\infty} \int f_j \, d\mu = B \qquad \text{(monotonicity)} \end{aligned}$$

■

Lemma. Let $f, g \geq 0$ be measurable and $\alpha \geq 0$. Then,

$$\int (f + \alpha g) \, d\mu = \int f \, d\mu + \alpha \int g \, d\mu$$

Proof. Recall from previous lemma we can always have simple function sequences $s_j \to f, u_j \to g$. By linearity of integral for simple functions,

$$s_1 + \alpha u_1 \leq s_2 + \alpha u_2 \leq \cdots \leq f + \alpha g$$

So that $\int (s_j + \alpha u_j)\, d\mu \to \int (f + \alpha g)\, d\mu$ by MCT. ■

Lemma. Let $f_j \geq 0$ be measurable. Then

$$\int \sum_{j=1}^{\infty} f_j \, d\mu = \sum_{j=1}^{\infty} \int f_j \, d\mu$$

Proof. Define $g_N := \sum_{j=1}^{N} f_j$

$$\begin{aligned} \int \sum_{j=1}^{\infty} f_j \, d\mu = \int \lim_{N\to\infty} &\overset{MCT}{=} \lim_{N\to\infty} \int g_N \, d\mu \\ &= \lim_{N\to\infty} \sum_{j=1}^{N} \int f_j \, d\mu \\ &= \sum_{j=1}^{\infty} \int f_j \, d\mu \end{aligned}$$

■

Lemma. Let $f \geq 0$ be measurable, then for $A \in \mathcal{F}$

$$A \mapsto \int_A f \, d\mu$$

is a measure.

Proof. See HW3. Consider the lemma. ■

Theorem. [Fatou's]. Let $f_j \geq 0$ be a sequence of measurable functions. Then,

$$\int \liminf_{j\to\infty} f_j \, d\mu \leq \liminf_{j\to\infty} \int f_j \, d\mu$$

Proof. Recall that $\liminf$ always holds for measurable f_j. ■

Proof.

$$\begin{aligned}
\int \liminf_{j\to\infty} f_j &\overset{\text{def}}{=} \int \lim_{k\to\infty} \overbrace{\inf_{j\geq k} f_j}^{:=g_k,\text{increasing}} \, d\mu \\
&\overset{\text{MCT}}{=} \lim_{k\to\infty} \int \underbrace{\inf_{j\geq k} f_j}_{\leq f_k} \, d\mu \\
&= \liminf_{k\to\infty} \int \underbrace{\inf_{j\geq k} f_j}_{\leq f_k} \, d\mu \\
&\leq \liminf_{k\to\infty} \int f_k \, d\mu
\end{aligned}$$

■

Useful Properties.

1. If $f \geq 0$ and $\int f \, d\mu = 0$, then $f = 0$ μ-a.e.
2. If $f = g$ μ-a.e., then $\int f \, d\mu = \int g \, d\mu$.
3. Let $f \geq 0$ be measruable with $\int f \, d\mu < \infty$ and $f(x) < \infty$, then, $f(x) < \infty$ μ-a.e.

Proof. (1). Notice that $\{f > 0\} = \bigcup_{j=1}^{\infty} \{f > \frac{1}{j}\}$. If $\mu(\{f > 0\}) > 0$, then $\exists j$ where $\mu(\{f > 1/j\}) > 0$. By contradiction, we have

$$0 = \int f \, d\mu \geq \int_{\{f>1/j\}} f \, d\mu \geq \int_{\{f>1/j\}} \frac{1}{j} \, d\mu = \frac{1}{j} \mu(\{f > 1/j\}) > 0$$

(2). Follows as $\int_A f \, d\mu = 0$ if $\mu(A) = 0$.

(3). We need $\mu(\{f = \infty\}) = 0$. Indeed,

$$\mu(\{f = \infty\}) \leq \mu(\{f \geq j\}) = \frac{1}{j} \int 1_{\{f\geq j\}} \, d\mu \leq \frac{1}{j} \int_{\{1\leq j\}} f \, d\mu \leq \frac{1}{j} \int f \, d\mu \to 0$$

■

Remark. Often to show something is finite/ 0, we can show its integral is finite/0.

4.3 The Integral of a General Measureable Function.

Notice that if $f : X \to \dot{\mathbb{R}}$, we can write $f^+, f^- \geq 0$ measurable where

$$f := f^+ - f^-, \quad \text{where} f^+ := \max(f, 0) = \frac{1}{2}(|f| + f), f^- := -\min(f, 0) = \frac{1}{2}(|f| - f)$$

Thus, we can define

$$\int f\,d\mu := \int f^+\,d\mu - \int f^-\,d\mu$$

Note that problem can arise when they are both ∞. Therefore, we will make this defintiion for the so-called **integrable functions** ($L^1(\mu)$) - those that satisfy

$$\int |f|\,d\mu < \infty \iff \int f^+\,d\mu, \int f^-\,d\mu < \infty$$

Caution: When using $\int f\,d\mu$, it is crucial to check the property above.

Remark.

1. f integrable on E means that $1_E f$ is $L^1(\mu)$.
2. If f is integrable,

$$\begin{aligned} |\int f\,d\mu| &= |\int f^+\,d\mu - \int f^-\,d\mu| \leq \int f^+\,d\mu + \int f^-\,d\mu \\ &= \int (f^+ + f^-)\,d\mu = \int |f|\,d\mu \end{aligned}$$

Proposition. If $f, g : X \to \dot{\mathbb{R}}$ is integrable and $\alpha \in \mathbb{R}$, then $f + \alpha g \in L^1(\mu)$ and

$$\int (f + \alpha g)\,d\mu = \int f\,d\mu + \alpha \int g\,d\mu$$

Proof. Define $h := f + \alpha g$. We have

$$\int |h|\,d\mu \leq \int (|f| + \alpha|g|)\,d\mu = \int |f|\,d\mu + \alpha \int |g|\,d\mu < \infty$$

Also we let $\alpha \geq 0$ first. Then

$$\begin{aligned} h^+ - h^- = h &= f + \alpha g \\ &= f^+ - f^- + \alpha g^+ - \alpha g^- \\ &\implies h^+ + f^- + \alpha g^- = h^- + f^+ + \alpha g^+ \\ &\implies \int h^+ + f^- + \alpha g^-\,d\mu = \int h^- + f^+ + \alpha g^+\,d\mu \\ &\implies \int h^+\,d\mu + \int f^- + \alpha \int f^-\,d\mu = ... \end{aligned}$$

Rearrange and get the desired result. Similar follows for $\alpha < 0$. ■

Note that monotonicity also holds for $f \leq g \implies \int f\,d\mu \leq \int g\,d\mu$. We have that

$$\int g\,d\mu = \int f\,d\mu + \int (g - f)\,d\mu \geq f$$

Lemma. Let E_j be measurable and disjoint sets, and $E := \bigcup_{i=1}^{\infty} E_j$.

1. If f is integrable over E, then f is also integrable on $E_j \forall j$, and

$$\int_E f\,d\mu = \sum_{i=1}^{\infty} \int_{E_j} f\,d\mu$$

2. Conversely, if f is integrable over E_j $\forall j$ AND $\sum_{i=1}^{\infty} \int_{E_j} |f|\,d\mu < \infty$, then f is integrable over E, and

$$\int_E f\,d\mu = \sum_{j=1}^{\infty} \int_{E_j} f\,d\mu$$

Proof. Exercise. ■

Theorem. [Dominated Convergence Theorem].
Let $f_j : X \to \dot{\mathbb{R}}$ be measruable with $|f_j| \leq g, \mu$-a.e, for some $g \in L^1(\mu)$. Assume also $f_j(x) \to f(x)$ μ-a.e. Then, f, f_j integrable, and

$$\int |f - f_j|\,d\mu \longrightarrow 0$$

In particular,

$$\lim_{j\to\infty} \int f_j\,d\mu = \int \lim_{j\to\infty} f_j\,d\mu = \int f\,d\mu$$

Proof. Notice that the set

$$N := \{f_j \text{ doesn't converge to } f\} \cup \bigcup_{j=1}^{\infty} \{|f_j| > g\}$$

satisfies $|N| = 0$, and $|f_j| \leq g \forall j$ in $X \setminus N$ and $|f| = \lim_{j\to\infty} |f_j| \leq g$ in $X \setminus N$. By modifying functions in a set of measure zero, WLOG $|f_j| \leq g$, $f_j \to f$ everywhere.

Now, the main idea of the proof is to use Fatou, so somehow we want non-negative functions. Notice that $|f - f_j| \leq |f| + |f_j| \leq 2g$. So, $h_j := 2g - |f - f_j| \geq 0$. Also $h_j \to 2g$ pointwise. So,

$$\begin{aligned}
\int 2g\,d\mu &= \int lim h_j\,d\mu \\
&\leq \liminf \int h_j\,d\mu && \text{(Fatou)} \\
&= \liminf \left(\int 2g\,d\mu - \int |f - f_j|\,d\mu \right) \\
&= \int 2g\,d\mu + \liminf \left(- \int |f - f_j|\,d\mu \right) && \text{(Think Carefully)} \\
&= \int 2g\,d\mu + \limsup \left(\int |f - f_j|\,d\mu \right) && \implies \limsup \int |f - f_j|\,d\mu \leq 0
\end{aligned}$$

Since

$$0 \le \liminf \int |f - f_j| \, d\mu \le \limsup \int |f - f_j| \, d\mu \le 0$$

$\exists \lim \int |f - f_j| \, d\mu = 0$. Then also,

$$| \int f_j \, d\mu - \int f \, d\mu| = | \int (f_j - f) \, d\mu| \le \int |f_j - f| \, d\mu \to 0$$

■

Example. Suppose $\varphi_j : X \to \mathbb{R}$ measurable with $\sum \int |\varphi_j| \, d\mu < \infty$. Then

$$\int (\sum |\varphi_j|) \, d\mu < \infty \implies \sum |\varphi_j(x)| < \infty \text{ a.e.}$$

So if $E := \{\sum |\varphi_j| = \infty\}$. Then, $\mu(E) = 0$ and $f(x) := \sum \varphi_j(x)$ converges absolutely on $X \setminus E$. Say, $f = 0$ on E. We want $\int f \, d\mu = \sum \int \varphi_i \, d\mu$.

To apply DCT, set $f_j(x) := \sum_{i=1}^{j} \varphi_i(x)$. Then, $f_j(x) \to f(x)$ a.e. .

$$\begin{aligned} \int f \, d\mu &= \int \lim f_j \, d\mu \\ &= \lim \int f_j \, d\mu && \text{(DCT)} \\ &= \lim_{j \to \infty} \sum_{i=1}^{j} \int \varphi_i \, d\mu = \sum_{i=1}^{\infty} \int \varphi_i \, d\mu && \text{(linearity)} \end{aligned}$$

In particular for DCT, we have

$$|f_j(x)| = | \sum_{i=1}^{\infty} \varphi_i(x)| \le \sum_{i=1}^{j} \varphi(x)| \le \sum_{i=1}^{\infty} |\varphi_i(x)| =: g$$

So g integrable, $\int g \, d\mu = \sum \int |\varphi_i| \, d\mu < \infty$ by assumption. So, DCT is ok.

4.4 Absolute Continuity of the Measures

Definition. Let μ, ν be measures on the same measruable space (X, Γ). We say ν is **absolute continuous** w.r.t. μ, denoted $\nu << \mu$, if $\mu(A) = 0 \implies \nu(A) = 0, \ A \in \Gamma$.

Stereotypical Example. Suppose μ is given and $f \ge 0$ fixed. $\nu(A) := \int_A f \, d\mu$. Then, $\nu << \mu$.

Lemma. Suppose μ, ν are measures on (X, Γ) and $\nu(X) < \infty$. Then $\nu << \mu$ if and only if $\forall \varepsilon > 0, \exists \delta > 0$ such that $\mu(A) < \delta$ implies $\nu(A) < \varepsilon$.

Proof. The $\Longleftarrow$ direction is *trivial*. For $\Longrightarrow$, assume $\nu << \mu$. Suppose by contradiction that $\exists \varepsilon > 0$ and sets $E_1, E_2, ...$ such that $\nu(E_i) < 2^{-i}$ but $\nu(E_i) \geq \varepsilon$. Define

$$A_k := \bigcup_{i=1}^{\infty} E_i, \quad A := \bigcup_{k=1}^{\infty} A_k$$

$\mu(A) = 0$, since $\forall k$ we have

$$\mu(A) \leq \mu(A_k) \leq \sum_{i=k}^{\infty} \mu(E_i) \leq \sum_{i=k}^{\infty} 2^{-i} \sim 2^{-k}$$

Note that we can $A \lesssim B$ if $A \lesssim cB$ for some not important c. $A \sim B$ if $A \lesssim B \lesssim A$. So, $\mu(A) = 0$.

[add small parts afterwards] ■

Corollary. In (X, Γ, μ), suppose $f : X \to \dot{R}$ is integrable. Then, $\forall \varepsilon > 0$, $\exists \delta > 0$ such that $\mu(A) < \delta \implies \int_A |f| \, d\mu < \varepsilon$.

4.5 Topology Review

Definition. A **topology** space is $(X, \mathcal{T})$ where

1. $\mathcal{T}$ is closed under arbitrary unions
2. $\mathcal{T}$ closed under finite intersections
3. $X, \emptyset \in \mathcal{T}$

Definition. [Not done yet from previous lecture]

Definition. A set $K \subset X$ is **compact** if every open cover has a finite subcover. If $K \subset \bigcup_\alpha V_\alpha$, $V_\alpha \subset X$ open, then $\exists \alpha_1, ..., \alpha_n$ such that $K \subset \bigcup_{i=1}^n V_i$. In particular, note the following results:

1. If $F \subset K$, F closed, K compact $\implies F$ compact.
2. If X Hausdorff, $K \subset X$ compact $\implies K$ closed.
3. $f : X \to Y$ for X, Y topological spaces, then $K \subset X \implies fK \subset Y$ compact.
4. $K_1 \cup K_2$ compact if K_i compact.
5. If X Hausdorff, $K_1, K_2 \subset X$ compact and $K_1 \cap K_2 = \emptyset \implies \exists$ neighborhood U_i of K_i such that $U_1 \cap U_2 = \emptyset$.
6. If X Hausdorff and there is a decreasing sequence $K_1 \subset K_2 \subset ...$ compact, then $K_i \neq \emptyset \forall i \implies \bigcap K_i \neq \emptyset$.

Definition. We say (X, τ) is a **locally compact** if $\forall x \in X$, $\exists$ neighborhood U of X such that $\overline{U}$ is compact.

Theorem. Let X be locally compact Hausdorff (LCH). If $K \subset X$ compact and U is a neighborhood of K, $\exists$ a neighborhood V of K such that

$$\overline{V} \subset U \text{ is compact}$$

Example. [Urysohn] Suppose there is compact K and open U with $K \subset U$. We want to make a compactly supported continuous function $f \approx 1_k$. So, pick neighborhood V of K such that $K \subset V \subset \overline{V} \subset U$, where $\overline{V}$ compact. Thus we can write

$$f(x) = \frac{dist(x, V^c)}{dist(x, K) + dist(x, V^c)}$$

where $x \mapsto dist(x, A)$ is continuous and lipschitz: $|d(x, A) - d(y, A)| \leq |x - y|$. Here, $f = 1$ on K, is continuous, in $[0, 1]$, and $f = 0$ on V^c. So, f supported on $\overline{V} \subset U$.

5 Riesz Representation Theorem

5.1 Topological Preliminaries

Definition. Let X be LCH. Then, we denote

$$C_c(X) = \{f : X \to \mathbb{R}; \text{continuous and } \operatorname{spt}(f) \text{ compact}\},$$

where the **support** is defined as

$$\operatorname{spt}(f) = \overline{\{x \in X : f(x) \neq 0\}}$$

Also, let $K, U \subset X$, $f : X \to \mathbb{R}$. We denote

$$K \prec f \iff \begin{cases} f \in C_c(X) \\ K \text{ compact} \\ 1_k \leq f \leq 1 \end{cases}, \text{ and } f \prec U \iff \begin{cases} f \in C_c(X) \\ U \text{ open} \\ 0 \leq f \leq 1 \\ \operatorname{spt}(f) \subset U \end{cases}$$

The Urysohn function f from example above satisfies $K \prec f \prec U$.

Theorem. **[Urysohn's Lemma.]** Let X be LCH, $K \subset U$, K compact, and U open. Then, $\exists f$ such that

$$K \prec f \prec U$$

Proof. Define $q_1 = 0, q_2 = 1$, and enumerate $(0, 1) \cap \mathbb{Q} = \{q_3, q_4, ...\}$. Using theorem mentionted previously twice, $\exists V_0, V_1$ such that $\overline{V_0}, \overline{V_1}$ compact and

$$K \subset V_1 \subset \overline{V_1} \subset V_0 \subset \overline{V_0} \subset U$$

By induction, suppose $n \geq 2$ and we have already chosen $V_{q_1}, V_{q_2}, ..., V_{q_n}$ such that $q_i < q_j \implies \overline{V_{q_j}} \subset \overline{V_{q_i}}$ and $\overline{V_{q_k}}$ compact $\forall k$. Now, we need to construct $V_{q_{n+1}}$. Let $q_i, q_j,\ i, j \in \{1, ..., n\}$ be the largest and smallest numbers such that $q_i < q_{n+1} < q_j$. Using the previous theorem again, $\exists V_{q_{n+1}}$ open such that $\overline{V_{q_{n+1}}}$ compact and

$$\overline{V}_{q_j} \subset V_{q_{n+1}} \subset \overline{V}_{q_{n+1}} \subset V_{q_j}$$

Proceeding by induction, by found collection of open sets indexed with q such that $K \subset V_1, \overline{V_0} \subset U$, each $\overline{V_q}$ compact, and $q > r$ implies $\overline{V_q} \subset V_r$. For each $q \in \mathbb{Q} \cap [0, 1]$, we define $f_q = q \cdot 1_{V_q}$.

Here, it is non-trivial that f is continuous. As V_q is open, $\{f_q > \alpha\}$ is open $\forall \alpha \in \mathbb{R}$. So, f_q is lower semicontinuous.

Also, $0 \leq f(x) \leq 1$. $\forall x \in K, 1 \geq f(x) \geq f_1(x) = \mathbb{I}_{V_q}(x) = 1$. Also, $f(x) = 0$ if $x \notin V_0$. So, $\operatorname{spt}(f) \subset \overline{V_0} \subset U$ compact.

For upper semicontinuity, define $g_r = 1_{\overline{V_r}} + r\mathbb{I}_{X \setminus \overline{V_r}}$, $f = \inf_g g_q$. Similarly for f, g is upper semi-continuous. Remains to show that $f = g$. First, $f_q(x) > g_r(x)$ only possible if $x \in V_q, q > r$ and $x \notin \overline{V}_r$, but $q > r \implies \overline{V}_q \subset V_r$, so $f(x) \leq g(x)$.

Also assume $f(x) < g(x)$. Pick rationals q, r such that $f(x) < r < q < g(x)$. But now $f(x) < r \implies x \notin V_r$. Also, $g(x) > q \implies x \notin X \setminus \overline{V}_q$. But, $q > r \implies \overline{V}_q \subset V_r$. ∎

Theorem. Let X be LCH, $V_1, ..., V_n \subset X$ open, and $K \subset V_1 \cup \cdots \cup V_n$ compact.Then, $\exists h_i \prec V_i$, $i = 1, ..., n$ such that $1_k \leq h_1 + \cdots + h_n \leq 1$.

Proof. If $x \in K$, $\exists i(x) = i$ such that $x \in V_i$ implies by theorem 1.15, $\exists$ neighborhood W_x of x such that $\overline{W_x} \subset V_1 = V_{i(x)}$ is compact. Notice $K \subset \bigcup_{x \in K} W_x \implies$ by compactness, $\exists x_1, ..., x_n \in K$ such that $K \subset \bigcup_{j=1}^m W_{x_j}$. Then, set $H_i = \bigcup_{j: i(x_j)=1} \overline{W_{x_j}} \subset V_i$, so H_i compact.

Apply Urysohn's to $H_i \subset V_i$, so $\exists g_i$ such that $H_i \prec g_i \prec V_i$. We want $h_1 + \cdots + h_n = 1 - \prod_{i=1}^n (1 - g_i)$. In fact, we can use

$$h_k := \left(\prod_{i=1}^{k-1} (1 - g_i) \right) g_k, \qquad k \in \{1, ..., n\}$$

Clearly, $h_k \prec V_k$. Now, we have $h_1 + h_2 = g_1 + (1 - g_1)g_2 = 1 - (1 - g_1)(1 - g_2)$ and proceed by induction. Now $h_1 + \cdots h_n = 1$ on K, since $K \subset H_1 \cup \cdots \cup H_n$ and $g_i = 1$ on H_i. ∎

5.2 Standard Version of Riesz Representation Theorem

For this discussion, $(X, \mathcal{F}, \mu)$ is a measure space such that X is LCH. Asuume $\mathcal{M}$ is Borel, i.e., $\operatorname{Bor}(X) \subset \mathcal{F}$. Assume also that μ is *locally finite*: $\mu(K) < \infty, \forall$ compact

$K \subset X$. Then, f is $\mathcal{F}$-measurable as $\{f > \alpha\} = f^{-1}(-\infty, \alpha)$ is open by continuity, so it is in $\mathrm{Bor}(X) \subset \mathcal{F}$.

Also, f is *integrable* as f supported means

$$\int |f| d\mu = \int_{\mathrm{spt}(f)} \underbrace{|f|}_{\lesssim 1} d\mu \lesssim \mu(K) < \infty$$

where $K := \mathrm{spt}(f)$. So, we can form $\Lambda : C_c(X \to \mathbb{R})$,

$$\Lambda(f) := \int f \, d\mu$$

Trivially, Λ is linear, as $\Lambda(\alpha_1 f_1 + \alpha_2 f_2) = \alpha_1 \Lambda f_1 + \alpha_2 \Lambda f_2$. Also, $f \geq 0 \implies \Lambda f \geq 0$, which we call Λ a "*positive linear functional*" in $C_c(X)$.

In particular, **Riesz Representation theorem** is the converse: all positive linear functionals look like this.

Theorem. [Riesz Representation Theorem.] Suppose X is LCH and $\Lambda : C_c(X) \to \mathbb{R}$ is an arbitary *positive linear functional* where

1. $\Lambda(\alpha_1 f_1 + \alpha_2 f_2) = \alpha_1 \Lambda(f_1) + \alpha_2 \Lambda(f_2)$ if $f_1, f_2 \in C_c(X)$ and $\alpha_1, \alpha_2 \in \mathbb{R}$.
2. $\Lambda f \geq 0$ if $f \in C_c(X)$ and $f \geq 0$.

Then, there exists a σ-algebra $\mathcal{F} \supset \mathrm{Bor}(X)$ and a *unique* measure $\mu : \mathcal{F} \to [0, \infty]$ such that $(X, \mathcal{F}, \mu)$ is complete, and

$$\Lambda(f) = \int f \, d\mu, \;\; f \in C_c(X)$$

where μ also satisfies the following properties:

1. μ locally finite; $\mu(K) < \infty, \forall$ compact $K \subset X$.
2. μ is outer regular. $\mu(E) = \inf\{\mu(V) : E \subset V, V \text{ open}\}, \forall E \in \mathcal{F}$
3. μ is inner regular for some sets: $\mu(E) = \sup\{\mu(K) : K \subset E, K \text{ compact}\}, \forall$ open E, and $\forall E \in \mathcal{F}$ with $\mu(E) < \infty$.

Proof. **Uniqueness.** Suppose $\mu_1, \mu_2 : \mathcal{F} \to [0, \infty]$ satisfies the three previous results and $\Lambda f = \int f \, d\mu, f \in C_c(X)$. First, $\mu_1(K) = \mu_2(K)$. If $K \subset X$ compact, use (2) to μ_2 so that for $\varepsilon > 0, \exists U \supset K$ open such that $\mu_2(U) \leq \mu_2(K) + \varepsilon$. By Urysohn, $\exists f$ such that $K \prec f \prec U$. So, $\mu_1(K) = \int \mathbb{I}_k \, d\mu_1 \leq \int f \, d\mu_1 = \Lambda f = \int f \, d\mu_2 \leq \mu_2(K) + \varepsilon$. As $\varepsilon \to 0$, $\mu_1(K) \leq \mu_2(K)$. By symmetry, $\mu_1(K) = \mu_2(K)$. Apply (3) to any open set U, so that $\mu_1(U) = \mu_2(U)$. By (2), $\mu_1(E) = \mu_2(E) \; \forall E \in \mathcal{F}$..

Construction of μ. Define for open sets $U \subset X$ that

$$\mu(U) := \sup\{\Lambda f : f \prec U\}$$

For $E \subset X$, define $\mu(E) = \inf\{\mu(U) : E \subset U, U \text{ open}\}$. Now, think of why the two definitions agree if $E = U$ open.

μ is an outer measure. Monotonicity is at least clear. For subadditivity, we want to prove $\mu(U_1 \cup U_2) \leq \mu(U_1) + \mu(U_2)$. If $U_1, U_2 \subset$ open, let $f \prec U_1 \cup U_2$. We need $\Lambda f \leq \mu(U_1) + \mu(U_2)$. Now let $K := \operatorname{spt} f$. As K compact and $K \subset U_1 \cup U_2$, $\exists$ partition of unity with $h_1 \prec U_1, h_2 \prec U_2$ with $h_1 + h_2 = 1$ on K. So on K, we can write

$$f = f \cdot \mathbb{I}_K = f(h_1 + h_2) = \underbrace{fh_1}_{\prec U_1} + \underbrace{fh_2}_{\prec U_2}$$

So, $\Lambda f = \Lambda(fh_1) + \Lambda(fh_2) \leq \mu(U_1) + \mu(U_2 z)$, and thus $\mu(U_1 \cup U_2) \leq \mu(U_1) + \mu(U_2)$.

Now, let $A_1, A_2, \ldots \subset X$ be arbitrary with $\mu(A_i) = \inf\{\mu(U) : U \supset A \text{ open}\}$. Let $\varepsilon > 0, \forall i$ choose open $U_1 \supset A_i$ such that $\mu(U_i) \leq \mu(A_i) + \varepsilon/2^i$. Set $U := \bigcup_{i=1}^\infty U_i, A := \bigcup_{i=1}^\infty A_i$ such that $A \subset U$, so $\mu(A) \leq \mu(U)$. We need that $\mu(U) \leq \sum_{i=1}^\infty \mu(A_i)$.

Here U is a countable, not finite union of open sets. Recall U open, so $\mu(U) = \sup\{\Lambda f : f \prec U\}$ and fix $f \prec U$. Let $K := \operatorname{spt} f$, then $\operatorname{spt} f \subset U = \bigcup_{i=1}^\infty U_i$. By compactness, $\exists n$ such that $K \subset U_1 \cup \cdots \cup U_n$. So $f \prec U_1 \cup \cdots \cup U_n$. So by definition,

$$\begin{aligned}
\Lambda f &\leq \mu(U_1 \cup \cdots \cup U_n) \\
&\leq \sum_{i=1}^{n} \mu(U_n) \\
&\leq \sum_{i=1}^{\infty} (\mu(A_i) + \varepsilon/2^i) \\
&= \sum_{i=1}^{\infty} \mu(A_i) + \varepsilon
\end{aligned}$$

So, we have the $\mu(U) \leq \sum_{i=1}^\infty \mu(A_i) + \varepsilon$.

Finally, for $\mu(\emptyset) = 0$, we will soon prove that $\mu(K) = \inf\{\Lambda f : K \prec f\} \forall$ compact $K \subset X$. In particular, with $K = \emptyset$, we have $\mu(\emptyset) = \inf\{\Lambda f : \emptyset \prec f\} \leq \Lambda 0 = 0$ as Λ is linear. So, μ is an outer measure.

Behavior of μ on compact sets. First, we prove $\mu(K) \leq \inf\{\Lambda f : K \prec f\}$. Fix f with $K \prec f$, and we with $\mu(K) \leq \Lambda f$. For $\delta \in (0,1)$, define $U_\delta := \{f > \delta\}$. Notice that $f = 1 > \delta$ on K, so $K \subset U_\delta$. Then $\mu(K) \leq \mu(U_\delta) = \sup\{\Lambda g : g \prec U_\delta\}$. As U_δ open, fix $g \prec U_\delta$. So, $g \leq \mathbb{I}_{U_\delta} \leq \frac{f}{\delta}$. Notice here there positivity of Λ means $g_1 \leq g_2 \implies \Lambda g_1 \leq \Lambda g_2$ for $g_1, g_2 \in C_c(X)$. So, we have that $\Lambda g \leq \Lambda\left(\frac{f}{g}\right) = \frac{1}{\delta}\Lambda f$. Now let $\delta \to 1$, so $\mu(K) \leq \Lambda f$ as desired.

For the converse inequality, we want to show $\inf\{\Lambda f : K \prec f\} \leq \mu(K)$. So, we need to produce an f so that $\Lambda f \leq \mu(K) + \varepsilon, \forall \varepsilon$. We choose an open $U \supset K$ such that $\mu(U) \leq \mu(K) + \varepsilon$, which can be found by definition of $\mu(K)$ with infimum. So by

Urysohn, $\exists f$ with $K \prec f \prec U$, so by definition $\Lambda f \leq \mu(U) \leq \mu(K) + \varepsilon$. Thus, We have that

$$\mu(K) = \inf\{\Lambda f : K \prec f\}$$

Notice that $\mu(K) < \infty$ follows from above, so μ *locally finite.*

Finite additivity. We seek to prove for disjoint compact K_i that

$$\mu(K_1 \cup \cdots \cup K_n) = \sum_{i=1}^{n} \mu(K_i)$$

Since subadditivity is already proved, we only need to consider the converse, $\mu(K_1 \cup K_2) \geq \mu(K_1) + \mu(K_2)$. By above, $\mu(K_1 \cup K_2) = \inf\{\Lambda g : K_1 \cup K_2 \prec g\}$. Fix $\varepsilon > 0$ and choose g such that $K_1 \cup K_2 \prec g$ and $\Lambda g \leq \mu(K_1 \cup K_2) + \varepsilon$. We will find f such that $K_1 \prec fg, K_2 \prec (1-f)g$. Once I have that, then $\mu(K_1)+\mu(K_2) \leq \Lambda(fg)+\Lambda((1-f)g) = \Lambda g \leq \mu(K_1 \cup K_2) + \varepsilon$.

It only remains to find such f. Since X LCH, $\exists U \supset K_1$ such that $U \cap K_2 = \emptyset$. Apply Urysohn to $K \subset U$, so $\exists f$ such that $K \prec f \prec U$. In particular, $f \equiv 1$ on K_1 and $f \equiv 0$ on K_2, so the above is satisfied.

Auxiliary Collection $\mathcal{F}_0$. We begin by building the corresponding $\sigma-$algebra $\mathcal{F}$. Define

$$\mathcal{F}_0 := \{E \subset X : \mu(E) < \infty\} \text{ and } \mu(E) = \sup\{\mu(K) : K \subset E \text{ compact}\}$$

Clearly, (1) $\mathcal{F}_0$ contains compact sets as $\mu(K) < \infty$. (2) $\mu(U) = \sup\{\mu(K) : K \subset U \text{ compact}\}$ if U open. In particular, $\{U \text{ open}; \mu(U) < \infty\} \subset \mathcal{F}_0$. So fix U open. Using $\mu(U) = \sup\{\Lambda f : f \prec U\}$, choose f with $f \prec U$ and $\Lambda f \geq \alpha$, where α can be any number satisfying $\alpha < \mu(U)$.

Write $K := \operatorname{spt} f \subset U$. We prove that $\Lambda f \leq \mu(K)$ as this implies $\alpha \leq \Lambda f \leq \mu(K)$. Suppose $V \supset K$ is open. Then, $f \prec V$. $\Lambda f \leq \mu(V) \implies \Lambda f \leq \inf\{\mu(V) : V \supset K \text{ open}\} = \mu(K)$

Countable Additivity in $\mathcal{F}_0$. Let $A_1, A_2, ... \in \mathcal{F}$ be disjoint. We prove that $\mu(A) = \sum_{i=1}^{\infty} \mu(A_i)$, $A = \bigcup_{i=1}^{\infty} A_i$. By subadditivity, it suffices to show $\mu(A) \geq \sum_{i=1}^{\infty} \mu(A_i)$. Let $\varepsilon > 0$, choose $K_i \subset A_i$ such that $\mu(A_i) \leq \mu(K_i) + \varepsilon/2^i$. (Notice K_i still disjoint since we approximate from inside.) For all n the following holds:

$$\begin{aligned}
\mu(A) &\geq \mu(K_1 \cup \cdots \cup K_n) \\
&= \sum_{i=1}^{n} \mu(K_i) \\
&\geq \sum_{i=1}^{n} \mu(A_i) - \sum_{i=1}^{n} \varepsilon/2^i = \sum_{i=1}^{n} \mu(A_i) - \varepsilon
\end{aligned}$$

Let $n \to \infty$ and $\varepsilon \to 0$, so $\mu(A) \geq \sum_{i=1}^{\infty} \mu(A_i)$. So additivity holds.

For later use, notice that this also gives "Fact 3 about $\mathcal{F}_0$", where for $A_1, A_2, ... \in \mathcal{F}_0, A = \cup A_i, \mu(A) < \infty \implies A \in \mathcal{F}_0$.

Regularity of $A \in \mathcal{F}_0$. If $A \in \mathcal{F}_0$ and $\varepsilon > 0, \exists$ compact $K \subset A$ and open $U \supset A$ such that $\mu(U \setminus K) < \varepsilon$.

First, choose open $U \supset A$ such that $\mu(U) < \mu(A) + \varepsilon/2$ (using outer measure). Then, choose compact $K \subset A$ such that $\mu(A) \leq \mu(K) + \varepsilon/2$ (using inner measure which holds in $\mathcal{F}_0$.)

We want to know $U \setminus K \in \mathcal{F}_0$. It is open as $U \setminus K = U \cap K^C$ where K closed. Also, $\mu(U \setminus K) \leq \mu(U) < \infty$. So fact 2 about $\mathcal{F}_0 \implies U \setminus K \in \mathcal{F}_0$. Now, $\mu(K) + \mu(U \setminus K) = \mu(U) < \mu(A) + \varepsilon/2 < \mu(K) + \varepsilon$, as $K, U \setminus K \in \mathcal{F}_0$ and by disjoint additivity in $\mathcal{F}_0$. So, $\mu(U \setminus K) < \varepsilon$.

$\mathcal{F}_0$ is closed under finite unions, finite intersections, and set differences. Suppose we have $A_1, A_2 \in \mathcal{F}_0$, and we want $A_1 \setminus A_2 \in \mathcal{F}_0$. This uses the previous step.

Let $\varepsilon > 0$. There $\exists K_i \subset A_i$ compact, and $\exists U_i \supset A_i$ open such that $\mu(U_i \setminus K_i) < \varepsilon$ as shown above. The goal is to approximate $A_1 \setminus A_2$ with a compact set. Turns out we can do $K_1 \setminus U_2$, which is a subset of $A_1 \setminus A_2$, also compact as closed subset of compact K. In particular,

$$\begin{aligned} A_1 \setminus A_2 &\subset U_1 \setminus K_2 \\ &\subset (U_1 \setminus K_1) \cup (K_1 \setminus K_2) \\ &\subset (U_1 \setminus K_1) \cup (K_1 \setminus U_2) \cup (U_2 \setminus K_2) \end{aligned}$$

So, as $(U_1 \setminus K_1), (U_2 \setminus K_2)$ can become arbitrarily small, $\mu(A_1 \setminus A_2) \leq 2\varepsilon + \mu(K_1 \setminus U_2)$, and thus $A_1 \setminus A_2 \in \mathcal{F}_0$.

Now, $A_1 \cup A_2 = (A_1 \setminus A_2) \cup A_2$, a disjoint union of elements in $\mathcal{F}_0$ with $\mu(A_1 \cup A_2) \leq \mu(A_1) + \mu(A_2) < \infty$. By fact 3, $A_1 \cup A_2 \in \mathcal{F}_0$.

Also, $A_1 \cap A_2 = \underbrace{A_2}_{\in \mathcal{F}_0} \setminus \underbrace{(A_1 \setminus A_2)}_{\in \mathcal{F}_0} \in \mathcal{F}_0$.

Definition of $\mathcal{F}$. We can define

$$\mathcal{F} = \{E \subset X : E \cap K \in \mathcal{F}_0 \; \forall \text{ compact } K \subset X\}$$

We claim that $\mathcal{F}$ is a σ-algebra.

1. $X \in \mathcal{F}$ since $X \cap K = K \in \mathcal{F}_0 \; \forall$ compact K.
2. Let $A \in \mathcal{F}, K$ compact. $(X \setminus A) \cap K = K \setminus (A \cap K) \in \mathcal{F}_0$. Therefore $X \setminus A \in \mathcal{F}$.
3. $A_1, A_2, ... \in \mathcal{F}$. Let $A := \cup A_i$ and K compact. Set $B_i := A_i \cap K, ...B_n = (A_n \cap K) \setminus (B_1 \cup \cdots \cup B_{n-1}), n \geq 2$. Now $A \cap K = \bigcup_{j=1}^{\infty} B_j$ where (B_j) disjoint, $B_j \in \mathcal{F}_0$. And $\mu(A \cap K) \leq \mu(K) < \infty \implies A \cap K \in \mathcal{F}_0$ by fact 3.

So, $\mathcal{F}$ is a σ-algebra.

Showing $\mathrm{Bor}(X) \subset \mathcal{F}$**.** Follows if $F \in \mathcal{F} \forall$ closed $F \subset X$. However, this is trivial as $F \cap K \in \mathcal{F}_0, \forall$ compact $K \subset X$, as $F \cap K$ closed and $\subset K$, so compact.

We claim that $\mathcal{F}_0 = \{A \in \mathcal{F}; \mu(A) < \infty\}$. Indeed, if $A \in \mathcal{F}_0$, then $\mu(A) < \infty$ by definition of $\mathcal{F}_0$ and $A \cap K \in \mathcal{F}_0 \forall$ compact $K \subset X$. So $\mathcal{F}_0 \subset \{A \in \mathcal{F} : \mu(A) < \infty\}$. Proof of converse: Let $A \in \mathcal{F}$ with $\mu(A) < \infty$. Need $\varepsilon > 0$, $\exists$ compact $K \subset A$ such that $\mu(A) \leq \mu(K) + \varepsilon$.

Choose any open $U \supset A$ with $\mu(U) < \infty$. We know $U \in \mathcal{F}_0$, and by fact 2, $\exists$ compact $H \subset U$ such that $\mu(U \setminus H) < \varepsilon/2$. By definition of $\mathcal{F}, H \cap A \in \mathcal{F}_0$, so $\exists$ compact $K \subset H \cap A$ such that $\mu(A \cap H) \leq \mu(K) + \varepsilon/2$. Thus

$$\begin{aligned} \mu(A) &\leq \mu(A \cap H) + \mu(A \setminus H) \\ &\leq \mu(K) + \varepsilon/2 + \varepsilon/2 \\ &\leq \mu(K) + \varepsilon \end{aligned}$$

So, $A \in \mathcal{F}_0$. So we have $\mathcal{F}_0 = \{A \in \mathcal{F} : \mu(A) < \infty\}$

Additivity of $\mathcal{F}$**.** Let $A_1, A_2, ... \in \mathcal{F}$ disjoint, $A := \cup A_i$. We want $\mu(A) = \sum \mu(A_i)$. This is trivial if $\mu(A_i) = \infty$ for some i (as $A_i \subset A$.) WLOG, $\mu(A_i) < \infty \forall i \implies A_i \in \mathcal{F}_0$. But additivity works in $\mathcal{F}_0$.

Completeness. Let $N \in \mathcal{F}$ with $\mu(N) = 0$ and $A \subset N$, we need $A \in \mathcal{F}$. Fix a compact K. Here $\mu(A \cap K) = 0$, and so $\mu(A \cap K) = \sup\{\mu(K') : K' \subset A \cap K, K' \text{ compact}\} = 0$ trivially. So, $A \cap K \in \mathcal{F}_0$.

Identity. We want to show $\Lambda f = \int f \, d\mu$. It is enough to show $\Lambda f \leq \int f \, d\mu$, since if true, then also $-\Lambda f = \Lambda(-f) \leq \int (-f) \, d\mu = -\int f \, d\mu \implies \Lambda f \geq \int f \, d\mu$.

Thus to show $\Lambda f \leq \int f \, d\mu, \forall f \in C_c(X)$, we fix $f \in C_c(X)$, let $[a, b]$ be an interval containing the image of f. Fix $\varepsilon > 0$. Divide $\tau_0 < a < \tau_1 < \cdots < \tau_n = b$ such that $\tau_{j+1} - \tau_j < \varepsilon$. Let $K_i = \operatorname{spt} f \subset X$. Write $K = \bigcup_j A_j$, where $A_j := K \cap f^{-1}(\tau_{j-1}, \tau_j]$, where A_j are *disjoint* borel sets.

Choose open $U_j' \supset A_j$ such that $\mu(U_j') \leq \mu(A_j) + \varepsilon/n$. Define $U_j := U_j' \cap f^{-1}(\tau_{j-1}, \tau_j + \varepsilon)$ and notice U_j is open, and $U_j \supset A_j \cap f^{-1}(\tau_{j-1}, \tau_j + \varepsilon) = A_j$ since A_j contains its intersection with $f^{-1}(\tau_{j-1}, \tau_j + \varepsilon)$. $\mu(U_j) \leq \mu(U_j') \leq \mu(A_j) + \varepsilon/n$, and $f(x) \leq \tau_j + \varepsilon$, $\forall x \in U_j$.

Now $K = \cup A_j \subset \bigcup U_j$, so there exists partition of unity $\{h_j\}_{j=1}^n$ subordinate to $\{U_j\}_{j=1}^n$ such that $h_j \prec U_j$ and $K \prec \sum_{j=1}^n h_j$. Now $f = \sum_{j=1}^n f h_j$ and thus we have $\Lambda f = \sum_{j=1}^n \Lambda(f h_j)$.

Recall that we derived that $\mu(K) = \inf\{\Lambda g : K \prec g\}$. So, $K \prec \sum_{j=1}^n h_j$ implies $\mu(K) \leq \Lambda\left(\sum_{j=1}^n h_j\right) = \sum_{j=1}^n \Lambda h_j$. Define the simple function $s = \sum_{j=1}^n (\tau_j - \varepsilon) \mathbb{I}_{A_j}$, where we defined to be $A_j = K \cap f^{-1}(\tau_{j-1}, \tau_j]$. Notice that if $x \in K, \exists! j$ such that $x \in A_j$, and then $s(x) = \tau_j - \varepsilon < \tau_{j-1} < f(x)$. So $s \leq f$. In particular,

$$\sum_{j=1}^{n} (\tau_j - \varepsilon)\mu(A_j) = \int s \, d\mu \leq \int f \, d\mu$$

Now,

$$\begin{aligned}
\Lambda f = \sum_{j=1}^{n} \Lambda(\underbrace{f h_j}_{\leq (\tau_j + \varepsilon) h_j}) &\leq \sum_{j=1}^{n} (\tau_j + \varepsilon) \Lambda h_j \\
&= \sum_{j=1}^{n} (\underbrace{|a| + \tau_j + \varepsilon}_{\geq |a| + \tau_0 + \varepsilon}) \Lambda h_j - |a| \underbrace{\sum_{i=1}^{n} \Lambda h_j}_{\geq \mu(K)} \\
&\leq \sum_{j=1}^{n} (|a| + \tau_j + \varepsilon) \Lambda h_j - |a| \mu(K)
\end{aligned}$$

Notice that $(|a| + \tau + \varepsilon)\Lambda h_j \leq (|a| + \tau_j + \varepsilon)(\mu(A_j) + \varepsilon/n)$ as $\Lambda h_j \leq \mu(U_j) \leq \mu(A_j) + \varepsilon/n$ following $h_j \prec U_j$. So we can write

$$\begin{aligned}
\Lambda f &\leq \sum_{j=1}^{n} (|a| + \tau_j + \varepsilon)(\mu(A_j) + \varepsilon/n) - |a|\mu(K) \\
&= \sum_{j=1}^{n} (|a| + \tau_j + \varepsilon)\mu(A_j) + \sum_{j=1}^{n} (\underbrace{|a| + \tau_j + \varepsilon}_{\leq (|a| + |b| + \varepsilon)\varepsilon}) \frac{\varepsilon}{n} - |a|\mu(K) \\
&\leq \left(\sum_{j=1}^{n} (\tau_j + \varepsilon)\mu(A_j) \right) + (|a| + |b| + \varepsilon)\varepsilon \\
&= \left(\sum_{j=1}^{n} (\tau_j + 2\varepsilon - \varepsilon)\mu(A_j) \right) + (|a| + |b| + \varepsilon)\varepsilon \\
&\leq \int f \, d\mu + 2\varepsilon\mu(K) + (|a| + |b| + \varepsilon)\varepsilon
\end{aligned}$$

■

Discussion. If we just look at the statement of the theorem, it doesn't intuitively remind us of the intermediate steps that comes along with the proof. We choose to formulate this second version of the Riesz theorem:

Theorem. [Riesz Representation Theorem, Restated.] Suppose X is LCH and $\Lambda : C_c(X) \to \mathbb{R}$ is a positive linear functional. Then, $\exists!$ *borel regular outer measure* $\mu : \mathcal{P}(X) \to [0, \infty]$ such that $\Lambda f = \int f \, d\mu (f \in C_c(X))$ AND

1. $\mu(K) < \infty \, \forall$ compact K
2. $\mu(E) = \inf\{\mu(V) : E \subset V, V \text{open}\}$ for **all** $E \subset X$.
3. $\mu(E) = \sup\{\mu(K) : K \subset E, K \text{ compact}\} \, \forall$ open E, and $\forall E \in \mathcal{M}_\mu$ with $\mu(E) < \infty$

There is an additional condition on X that can be assumed such that (3) can be stated $\forall E \in \mathcal{M}_\mu$.

Definition. A set $A \subset X$ is σ**-compact** if $A = \cup_{i=1}^{\infty} K_i, K_i$ compact.

Example. $\mathbb{R}^d = \cup_{i=1}^{\infty} \overline{B}(0, i)$.

Note. In the rest of the section, μ comes from Riesz.

Remark. Suppose A is σ-compact, $A \in \mathcal{M}_\mu$. If $\mu(A) < \infty$, inner regularity holds. Does it hold if $\mu(A) = \infty$? Yes, as $A = \bigcup K_i, K_1 \subset K_2 \subset \cdots$. So, $\infty = \mu(A) = \lim \mu(K_i) \implies \sup\{\mu(K) : K \subset A, K \text{ compact}\} = \infty = \mu(A)$, and inner regularity holds.

Lemma. Let X be LCH and σ-compact. Let μ be an outer measure as in Riesz, then

1. If $A \in \mathcal{M}_\mu(X)$, then $\forall \varepsilon \geq 0, \exists$ a closed set $F \subset A$, and an open set $U \supset A$ such that $\mu(U \setminus F) < \varepsilon$.
2. Inner Regularity holds $\forall A \in \mathcal{M}_\mu(X)$.
3. If $A \in \mathcal{M}_\mu(X)$, $\exists \mathcal{F}_\sigma$ set (countable union of closed sets, borel in particular) $H \subset A$ and a G_δ-set (countable intersection of open sets) $G \supset A$ such that $\mu(G \setminus H) = 0$

Proof. (1) Write $X = \bigcup_{i=1}^{N} K_i, K_i$ compact. Let $A \in \mathcal{M}_\mu(X)$. Notice that $A \cap K_i \in \mathcal{M}_\mu(X)$ since K_i borel, and $\mu(A \cap K_i) \leq \mu(K_i) < \infty$ using outer regularity. Let $U_i \supset A \cap K_i$ be open such that $\mu(U_i) < \mu(A \cap K_i) + \varepsilon/2^{i+1}$. This implies $\mu(U_i \setminus (A \cap K_i)) < \varepsilon/2^{i+1}$. since $U_i, A \cap K_i \in \mathcal{M}_\mu$ and $\mu(A \cap K_i) < \infty$. Let $U := \bigcup U_i \supset \bigcup (A \cap K_i) = A_j$, U open. Also, $\mu(U \setminus A) = \mu\left(\bigcup (U_i \setminus A)\right) \leq \sum \mu(U_i \setminus A) \leq \sum \mu(U_i \setminus (A \cap K_i)) < \varepsilon/2$. ■